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PERIODIC WAVE SOLUTIONS AND SOLITARY WAVE SOLUTIONS FOR THE NEW SOLITON EQUATION

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Abstract

Some exact travelling wave solutions had been found of new soliton equation. What are the dynamical behaviour of these travelling wave solutions and how do they depend on the parameters of the systems? This paper will answer these questions by using the method of dynamical systems. Three exact explicit parametric representations of the travelling wave solutions for new soliton equation are given.

1. Introduction

Solitons play an increasingly important role in nonlinear waves, dynamical systems, and analytical mechanics. It has been significant in soliton theory for us to find more new integrable systems.

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In 2007, Qiao [6] proposed the following new soliton equation:

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x. \tag{1}$$

This equation can be derived from the two-dimensional Euler equation by using the approximation procedure. The author has proved that Equation (1) has bi-Hamiltonian operators and Lax pairs through solving a crucial matrix equation. In addition, the author found the so-called W/M-shape peak solitons and one-single-peak soliton of Equation (1).

In this paper, we consider the following new soliton equation:

$$m_t = a\left(\frac{1}{m^2}\right)_{xxx} + b\left(\frac{1}{m^2}\right)_x. \tag{2}$$

Since the dynamical behaviour of the corresponding travelling wave systems of Equation (2) has not been studied before, we shall consider the dynamical bifurcation [1-3] of the travelling wave systems and obtain all possible exact explicit parametric representations of travelling wave solutions of Equation (2).

Let

$$m(x, t) = \phi(x - ct) = \phi(\xi), \quad \xi = x - ct. \tag{3}$$

Substituting Equation (3) into Equation (2) and integrating Equation (2) once with respect to ξ, we obtain

$$2a\phi^{-3}\phi_{\xi\xi} = 6a\phi^{-4}\phi_\xi^2 + b\phi^{-2} + c\phi, \tag{4}$$

where the integral constants are taken as zeros.

Equation (4) is equivalent to the planar dynamical system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{6ay^2 + b\phi^2 + c\phi^5}{2a\phi}. \tag{5}$$

Clearly, system (5) is a singular travelling wave system of the first type with the singular straight line $\phi = 0$ and the first integral

$$H(\phi, y) = y^2\phi^{-6} + \frac{b}{4a}\phi^{-4} + \frac{c}{a}\phi^{-1}. \tag{6}$$

To investigate the dynamics of the orbits of Equation (5), we first consider the associated regular system of Equation (5) as follows:

$$\frac{d\phi}{d\zeta} = 2ay\phi, \quad \frac{dy}{d\zeta} = 6ay^2 + b\phi^2 + c\phi^5, \tag{7}$$

where $d\xi = 2a\phi d\zeta$.

2. Bifurcations of Phase Portraits of System (7)

In this section, we first study the bifurcations of phase portraits of Equation (7).

It is easy to see that the origin $O(0, 0)$ and $E(\phi_1, 0)$ are two equilibrium points of Equation (7), where $\phi_1 = -\sqrt[3]{\frac{b}{c}}$.

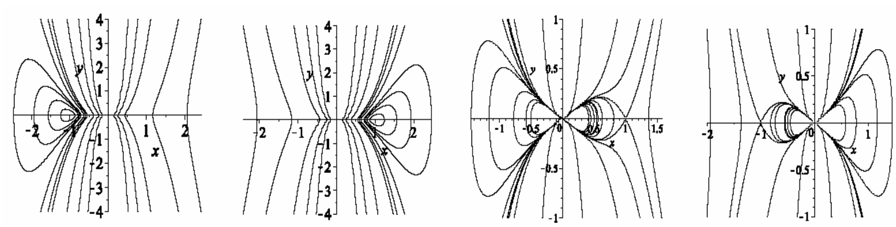
Let M be the coefficient matrix of the linearized system of Equation (7). We have

$$\det M(0, 0) = 0, \quad \det M(\phi_1, 0) = 6ab\phi_1^2.$$

Write that

$$h_0 = H(0, 0), \quad h_1 = H(\phi_1, 0).$$

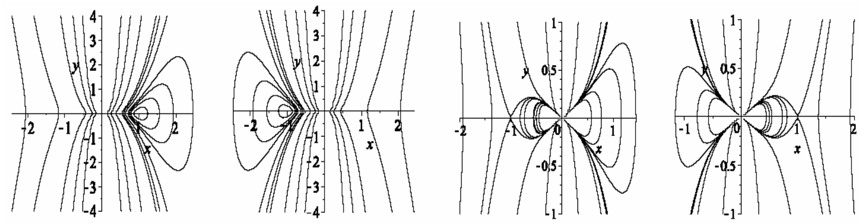
By the qualitative analysis, we know that for $c > 0$, the bifurcations of the phase portraits of Equation (7) are shown in Figure 1 ((1)-(4)).



(1) $a > 0, b > 0$ (2) $a < 0, b < 0$ (3) $a > 0, b < 0$ (4) $a < 0, b > 0$

Figure 1. The bifurcations of the phase portraits of Equation (7) for $c > 0$.

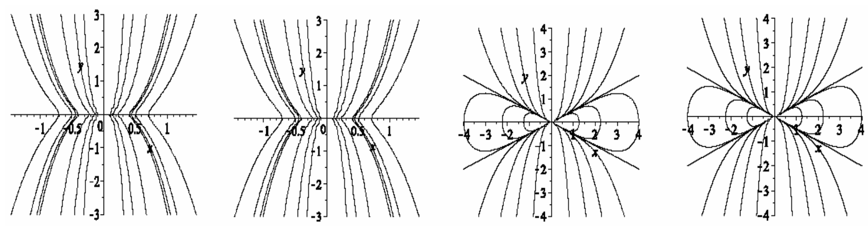
For $c < 0$, the bifurcations of the phase portraits of Equation (7) are shown in Figure 2 ((1)-(4)).



(1) $a > 0,\ b > 0$ (2) $a < 0,\ b < 0$ (3) $a > 0,\ b < 0$ (4) $a < 0,\ b > 0$

Figure 2. The bifurcations of the phase portraits of Equation (7) for $c < 0$.

For $c = 0$, the bifurcations of the phase portraits of Equation (7) are shown in Figure 3 ((1)-(4)).



(1) $a > 0,\ b > 0$ (2) $a < 0,\ b < 0$ (3) $a > 0,\ b < 0$ (4) $a < 0,\ b > 0$

Figure 3. The bifurcations of the phase portraits of Equation (7) for $c = 0$.

3. The Parametric Representations of Bounded Orbits Defined by $H(\phi, y) = h$ of Equation (5) when $c > 0$

In this section, we give the parametric representations for bounded orbits defined by $H(\phi, y) = h$ of Equation (5) in different parameter conditions.

We see from Equation (6) that

$$y^2 = \phi^2 \left(h\phi^4 - \frac{c}{a}\phi^3 - \frac{b}{4a}\right). \tag{8}$$

By using the first equation of Equation (5), we obtain

$$\int \frac{d\phi}{\phi\sqrt{h\phi^4 - \frac{c}{a}\phi^3 - \frac{b}{4a}}} = \xi. \tag{9}$$

It follows the parametric representations of the orbits defined by $H(\phi, y) = h$.

3.1. The case $a > 0$ and $b > 0$ (see Figure 1(1))

In this case, $h_1 < 0$, for every $h \to h_1^+$, corresponding level curve defined by $H(\phi, y) = h$ is a family of periodic orbits. Equation (8) can be written as

$$y^2 = |h|\phi^2(r_1 - \phi)(\phi - r_2)[(\phi - b_1)^2 + a_1^2].$$

Thus, we obtain a family of periodic solutions of Equation (5) as follows:

$$\phi(\omega) = \frac{r_2 A + r_1 B + (r_2 A - r_1 B) cn(\omega, k)}{A + B + (A - B) cn(\omega, k)},$$

$$\xi(\omega) = \frac{A + B}{\sqrt{|h|}(r_2 A - r_1 B)}\left(\alpha_1 \omega - \frac{\alpha(\alpha - \alpha_1)}{1 - \alpha^2}\sqrt{\frac{1 - \alpha^2}{k^2 + k_1^2 \alpha^2}} \arctan\left(\sqrt{\frac{k^2 + k_1^2\alpha^2}{1 - \alpha^2}} sd(\omega, k)\right)\right)$$

$$+ \frac{A + B}{\sqrt{|h|}(r_2 A - r_1 B)} \frac{\alpha - \alpha_1}{1 - \alpha^2} \Pi\left(\arccos(cn(\omega, k)), \frac{\alpha^2}{\alpha^2 - 1}, k\right), \tag{10}$$

where

$$\alpha = \frac{r_2 A - r_1 B}{r_2 A + r_1 B}, \quad \alpha_1 = \frac{A - B}{A + B}, \quad k^2 = \frac{(r_1 - r_2)^2 - (A - B)^2}{4AB},$$

$$k_1^2 = 1 - k^2, \quad A^2 = (r_1 - b_1)^2 + a_1^2, \quad B^2 = (r_2 - b_1)^2 + a_1^2.$$

3.2. The case $a < 0$ and $b < 0$ (see Figure 1(2))

For $h \to h_1^+ (h_1 < 0)$, the level curves defined by $H(\phi, y) = h$ are a family of periodic orbits. By using Equation (9), corresponding to the periodic orbit, we have the same parametric representations as Equation (10).

3.3. The case $a > 0$ and $b < 0$ (see Figure 1(3))

In this case, $h_1 > 0$, for every $h \to -h_1$, the level curves defined by $H(\phi, y) = h$ are two homoclinic orbits. Equation (8) can be written as

$$y^2 = |h|\phi^2(r_1 - \phi)(\phi - r_2)[(\phi - b_1)^2 + a_1^2].$$

By using Equation (9), corresponding to the homoclinic orbit in the left phase plane, we have

$$\phi(\omega) = \frac{r_2 A + r_1 B + (r_2 A - r_1 B) cn(\omega, k)}{A + B + (A - B) cn(\omega, k)},$$

$$\xi(\omega) = \frac{A + B}{\sqrt{|h|}(r_2 A - r_1 B)}\left(\alpha_1 \omega + \frac{\alpha(\alpha - \alpha_1)}{2\sqrt{(\alpha^2 - 1)(k^2 + k_1^2\alpha^2)}}\right.$$

$$\left. \times \ln \frac{\sqrt{k^2 + k_1^2\alpha^2}\, dn(\omega, k) + \sqrt{\alpha^2 - 1}\, sn(\omega, k)}{\sqrt{k^2 + k_1^2\alpha^2}\, dn(\omega, k) - \sqrt{\alpha^2 - 1}\, sn(\omega, k)}\right)$$

$$+ \frac{A + B}{\sqrt{|h|}(r_2 A - r_1 B)} \frac{\alpha - \alpha_1}{1 - \alpha^2} \Pi\left(\arccos(cn(\omega, k)), \frac{\alpha^2}{\alpha^2 - 1}, k\right), \quad (11)$$

where

$$\alpha = \frac{r_2 A - r_1 B}{r_2 A + r_1 B}, \quad \alpha_1 = \frac{A - B}{A + B}, \quad k^2 = \frac{(r_1 - r_2)^2 - (A - B)^2}{4AB},$$

$$k_1^2 = 1 - k^2, \quad A^2 = (r_1 - b_1)^2 + a_1^2, \quad B^2 = (r_2 - b_1)^2 + a_1^2.$$

3.4. The case $a < 0$ and $b > 0$ (see Figure 1(4))

(1) In this case, $h_1 > 0$, for every $h \to h_1^-$, the level curves defined by $H(\phi, y) = h$ are a family of homoclinic orbits. Equation (8) can be written as

$$y^2 = h\phi^2(\phi - r_1)(\phi - r_2)[(\phi - b_1)^2 + a_1^2].$$

By using Equation (9), we have

$$\phi(\omega) = \frac{r_2 A - r_1 B - (r_2 A + r_1 B)cn(\omega, k)}{A - B - (A + B)cn(\omega, k)},$$

$$\xi(\omega) = \frac{(B - A)g}{\sqrt{h}(r_2 A + r_1 B)}\left(\alpha_2\omega + \frac{\alpha(\alpha - \alpha_2)}{2\sqrt{(\alpha^2 - 1)(k^2 + k_1^2\alpha^2)}}\right.$$

$$\left.\times \ln\frac{\sqrt{k^2 + k_1^2\alpha^2}dn(\omega, k) + \sqrt{\alpha^2 - 1}\,sn(\omega, k)}{\sqrt{k^2 + k_1^2\alpha^2}dn(\omega, k) - \sqrt{\alpha^2 - 1}\,sn(\omega, k)}\right)$$

$$+ \frac{(B - A)g}{\sqrt{h}(r_2 A + r_1 B)}\frac{\alpha - \alpha_2}{1 - \alpha^2}\Pi\left(\arccos(cn(\omega, k)), \frac{\alpha^2}{\alpha^2 - 1}, k\right), \quad (12)$$

where

$$\alpha = \frac{r_2 A + r_1 B}{r_1 B - r_2 A}, \quad \alpha_2 = \frac{A + B}{B - A}, \quad k^2 = \frac{(A + B)^2 - (r_1 - r_2)^2}{4AB},$$

$$k_1^2 = 1 - k^2, \quad A^2 = (r_1 - b_1)^2 + a_1^2, \quad B^2 = (r_2 - b_1)^2 + a_1^2, \; g = \frac{1}{\sqrt{AB}}.$$

(2) For $h \to -h_1$, the level curves defined by $H(\phi, y) = h$ are two homoclinic orbits. Equation (8) can be written as

$$y^2 = |h|\phi^2(r_1 - \phi)(\phi - r_2)[(\phi - b_1)^2 + a_1^2].$$

By using Equation (9), corresponding to the homoclinic orbit in the left phase plane, we have the same parametric representations as Equation (11).

4. The Parametric Representations of Bounded Orbits Defined by $H(\phi, y) = h$ of Equation (5) when $c < 0$

4.1. The case $a > 0$ and $b > 0$ (see Figure 2(1))

For $h \to h_1^+ (h_1 < 0)$, the level curves defined by $H(\phi, y) = h$ are a family of periodic orbits. By using Equation (9), corresponding to the periodic orbit, we have the same parametric representations as Equation (10).

4.2. The case $a < 0$ and $b < 0$ (see Figure 2(2))

For $h \to h_1^+ (h_1 < 0)$, the level curves defined by $H(\phi, y) = h$ are a family of periodic orbits. By using Equation (9), corresponding to the periodic orbit, we have the same parametric representations as Equation (10).

4.3. The case $a > 0$ and $b < 0$ (see Figure 2(3))

(1) In this case, $h_1 > 0$, for every $h \to -h_1$, the level curves defined by $H(\phi, y) = h$ are two homoclinic orbits. By using Equation (9), corresponding to the homoclinic orbit in the left phase plane, we have the same parametric representations as Equation (11).

(2) For every $h \to h_1^-$, the level curves defined by $H(\phi, y) = h$ are a family of homoclinic orbits. By using Equation (9), we have the same parametric representations as Equation (12).

4.4. The case $a < 0$ and $b > 0$ (see Figure 2(4))

In this case, $h_1 > 0$, for every $h \to -h_1$, the level curves defined by $H(\phi, y) = h$ are two homoclinic orbits. By using Equation (9), corresponding to the homoclinic orbit in the left phase plane, we have the same parametric representations as Equation (11).

5. The Parametric Representations of Bounded Orbits Defined by $H(\phi, y) = h$ of Equation (5) when $c = 0$

The case $a > 0, b < 0$ (see Figure 3(3)) and the case $a < 0, b > 0$ (see Figure 3(4)).

In these cases, for $h \in (-\infty, 0)$, the level curves defined by $H(\phi, y) = h$ are two homoclinic orbits. By using Equation (9), corresponding to the homoclinic orbit in the left phase plane, we have the same parametric representations as Equation (11).

To sum up, under different parameter conditions, we obtain three exact parametric representations of $\phi(\xi)$ given by Equations (10)-(12).

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